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DEGRADABLE SYSTEMS: A SURVEY OF MULTISTATE SYSTEM
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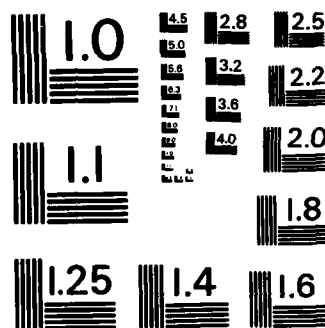
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ITEM #20, CONTINUED: > We present a survey of recent papers which treat the more sophisticated and more realistic models in which components and systems may assume many states ranging from perfect functioning to complete failure. Our survey updates and complements a previous survey by El-Neweihi and Proschan (1978). Some new results are included.

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**Degradable Systems: A Survey of
Multistate System Theory**

by

Emad El-Neweihi¹ and Frank Proschan²

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**The Florida State University
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Abstract

The vast majority of reliability analyses assume that components and system are in either of two states: functioning or failed. However, in many real life situations we are actually able to distinguish among various "levels of performance" for both system and components. For such situations, the existing dichotomous model is a gross oversimplification and so models assuming degradable (multistate) systems and components are preferable since they are closer to reality.

We present a survey of recent papers which treat the more sophisticated and more realistic models in which components and systems may assume many states ranging from perfect functioning to complete failure. Our survey updates and complements a previous survey by El-Newehi and Proschan (1978). Some new results are included.

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MATTHEW J. KATLER
Chief, Technical Information Division

1. Introduction

The theory of binary coherent systems serves as a unifying foundation for a mathematical and statistical theory of reliability. In this theory systems and components are assumed to be in one of two states: functioning or failed. In many real-life situations, however, the systems and their components are actually capable of assuming a whole range of levels of performance, varying from perfect functioning to complete failure. In order to describe more adequately the performance of such "degradable" systems and components, researchers felt the need to develop the theory of multistate coherent systems.

Until recently, little work had been done on this more general problem of multistate systems. However, a growing interest in this area is indicated by the increasing number of research papers currently being written on this subject. In this paper a survey is made of recent treatments of multistate models performed by Barlow and Wu [2], Block and Savits [3], [4], Borges and Rodrigues [5], El-Newehi, Proschan and Sethuraman [6], El-Newehi [8], Griffith [9], Griffith and Govindarajulu [10], Natvig [13] and Ross [15]. This survey updates and complements a previous one by El-Newehi and Proschan [7].

We now summarize the contents of this paper. Our formulation and treatment are similar to that of Barlow and Proschan [1] for the binary case. In Section 2 we present the notation and terminology used throughout the paper. In Section 3 deterministic models of multistate systems are presented. For the system and for each of its components we distinguish among different "levels of performance" represented by the elements of a totally ordered set S called the state space. The vector

$\underline{x} = (x_1, \dots, x_n)$ representing the states of the n components takes its values in S^n , where S^n is the n^{th} Cartesian power of S . The state of the system is represented by a function $\phi: S^n \longrightarrow S$ of component states. Set-theoretic and axiomatic approaches are adopted by various authors to introduce a variety of classes of multistate systems. We survey the different models and their structural properties, occasionally comparing and contrasting them.

In Section 4 we investigate the probabilistic aspects of multistate models. The random vector $\underline{X} \equiv (X_1, \dots, X_n)$ represents the states of the n components and the random variable $\phi(\underline{X})$ the state of the system. We survey the relationship between the stochastic performance of the system and the stochastic performance of its components. When the exact values of system performance probabilities are difficult to compute, bounds are provided.

Finally in Section 5 we survey dynamic aspects of degradable systems. At time 0, the system and each of its components are at the maximal level of performance. As time passes, the performance levels of components (and consequently of the system) deteriorate to lower levels until finally system level 0 (complete failure) is reached. Classes of 1-dimensional decreasing stochastic processes generalizing known classes of life distributions are presented. Multidimensional versions of such classes suitable for describing dependent components are surveyed. Generalized IFRA and NBU closure theorems are presented.

Results from the literature summarized in this survey have been credited to the researcher(s) responsible. However, there are some results in the present paper which are new and due to us. For such results we do not specify authorship.

We have used the term "degradable" as synonymous with "multistate". Our purpose in introducing the term is to help bridge the gap between the reliability theorist and the reliability practitioner. The theorist uses "multistate", the practitioner uses "degradable". We believe strongly that future growth and application of a rich multistate theory and its practical application to degradable systems will be a consequence of continued interaction between theorist and practitioner. A good example of practical interest is contained in Govindarajulu and Griffith [10].

2. Notation, Definitions, and Terminology.

The vector $\underline{x} \equiv (x_1, \dots, x_n)$ denotes the vector of states of components $1, \dots, n$.

$C = \{1, \dots, n\}$ denotes the set of component indices.

$(j_i, \underline{x}) \equiv (x_1, \dots, x_{i-1}, j, x_{i+1}, \dots, x_n)$, where $j = 0, 1, \dots, M$.

$(\cdot_i, \underline{x}) \equiv (x_1, \dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_n)$.

$\underline{j} \equiv (j, \dots, j)$.

$\underline{y} \leq \underline{x}$ means that $y_i \leq x_i$, $i = 1, \dots, n$.

$\underline{y} < \underline{x}$ means that $y_i \leq x_i$, $i = 1, \dots, n$, and $y_i < x_i$ for some i .

$\underline{\alpha} = (\alpha_0, \alpha_1, \dots, \alpha_M)$ is a probability vector means that $\alpha_j \geq 0$,

$j = 0, \dots, M$ and $\sum_{j=0}^M \alpha_j = 1$.

For probability vectors $\underline{\alpha}$ and $\underline{\alpha'}$, $\underline{\alpha} \leq^{\text{st}} \underline{\alpha'}$ means that

$$\sum_{j=\ell}^M \alpha_j \leq \sum_{j=\ell}^M \alpha'_j, \quad \ell = 0, 1, \dots, M.$$

A subset $U \subset R^n$ is an upper set if $\underline{x} \in U$ and $\underline{x} \leq \underline{y}$ imply that $\underline{y} \in U$.

A subset $L \subset R^n$ is a lower set if $\underline{x} \in L$ and $\underline{y} \leq \underline{x}$ imply $\underline{y} \in L$.

$$x \vee y \equiv \max(x, y).$$

$$\underline{x} \vee \underline{y} \equiv (x_1 \vee y_1, \dots, x_n \vee y_n).$$

$$x \wedge y \equiv \min(x, y).$$

$$\underline{x} \wedge \underline{y} \equiv (x_1 \wedge y_1, \dots, x_n \wedge y_n).$$

"Increasing" is used in place of "nondecreasing" and "decreasing" is used in place of "nonincreasing". When we say $f(x_1, \dots, x_n)$ is increasing we mean f is increasing in each argument.

Given a univariate distribution F , its complement $1-F$ is denoted by \bar{F} .

Given a set S , S^n denotes its n^{th} Cartesian power. R denotes the set of real numbers.

3. Deterministic Models for Multistate Systems.

First let us recall the definition of a binary coherent system of n components. The vector $\underline{x} = (x_1, \dots, x_n)$ represents the states of the n components where x_i is either 0 or 1, $i = 1, \dots, n$. The state of the system is determined by a structure function $\phi: \{0, 1\}^n \longrightarrow \{0, 1\}$. The structure function ϕ satisfies certain conditions that represent intuitively reasonable properties of systems encountered in practice. The following two conditions are required for a binary system to be a coherent structure [1, Def. 2.1, p. 6]:

- (i) The function $\phi(\underline{x})$ is increasing.
- (ii) For each i there exists a vector (\cdot, \underline{x}) such that $\phi(1_i, \underline{x}) > \phi(0_i, \underline{x})$. This means that the function ϕ is not constant in any of its arguments.

Condition (i) expresses the reasonable assumption that improving component performance should not degrade system performance. Condition

(ii) asserts that each component is relevant to system performance, thus eliminating from consideration components which have no effect on system performance. It follows from (i) and (ii) that

$$(iii) \quad \phi(\underline{1}) = 1 \text{ and } \phi(\underline{0}) = 0.$$

The class of binary coherent structures is precisely the class of all structure functions ϕ that have the following representation:

$$\phi(\underline{x}) = \max_{1 \leq j \leq r} \min_{i \in P_j} x_i \quad \text{for all } \underline{x} \in \{0,1\}^n,$$

where P_1, P_2, \dots, P_r are nonempty subsets of $\{1, \dots, n\}$ such that

$$\bigcup_{j=1}^r P_j = \{1, \dots, n\} \text{ and } P_i \not\supset P_j \text{ for } i \neq j. \text{ The sets } P_1, \dots, P_r \text{ are called}$$

the min path sets of the structure function ϕ (a dual representation in terms of "min cut" sets is also possible). Thus the same class of binary coherent structures can be obtained via either the axiomatic approach or the set-theoretic approach.

The binary model however, is an oversimplification in describing a situation in which both the system and its components are capable of assuming a whole range of levels of performance, varying from perfect functioning to complete failure. For such a case, a larger state space S is needed to describe the situation more adequately. Also useful definitions for multistate structure functions must be provided to relate the performance of the system to the performance of its components. A theory of multistate structures can then serve as a unifying foundation for a mathematical and statistical theory of reliability in the multistate case. Most of the earlier treatments dealing with multistate situations investigate only very special applications without trying to build a general framework for a multistate theory. (See for example Hirsch et al [11] and Postelnicu [14].)

More recent and more comprehensive research on multistate systems is performed by Barlow and Wu [2], Block and Savits [4], Borges and Rodrigues [5], El-Newehi, Proschan, and Sethuraman [6] (hereafter referred to as EPS [6]), El-Newehi and Proschan [7], Griffith [9], Natvig [13], and Ross [15]. Two approaches are adopted by these authors to introduce their classes of multistate structures: the set-theoretic approach and the axiomatic approach.

The set-theoretic approach is followed by Barlow and Wu [2] who introduce a class of multistate structure functions based on the concept of min path (min cut) sets of binary coherent systems. Consider a system of n components. Assume that the state space for each of the components as well as for the system is the set $S = \{0, 1, \dots, M\}$ where 0 denotes the failed state and M denotes the perfect state. Let P_1, \dots, P_r be nonempty subsets of C such that $\bigcup_{j=1}^r P_j = C$ and $P_i \not\supset P_j, i \neq j$.

The structure function $\phi: S^n \longrightarrow S$ is defined by

$$\phi(\underline{x}) = \max_{1 \leq j \leq r} \min_{i \in P_j} x_i, \quad (3.1)$$

where $\underline{x} \in S^n$ represents the states of components $1, 2, \dots, n$. Let ϕ' be the binary coherent structure function whose min path sets are P_1, \dots, P_r . The multistate coherent structure ϕ specified in (3.1) can then be expressed in terms of ϕ' as follows: For each $i = 1, \dots, n$, let

$$y_{ij} = \begin{cases} 1 & \text{if } x_i \geq j \\ 0 & \text{o.w.} \end{cases}$$

and let $\underline{y}_j = (y_{1j}, \dots, y_{nj})$, $j = 0, 1, \dots, M$. It is easy to see that $\phi(\underline{x}) \geq j$ iff $\phi'(\underline{y}_j) = 1$, and

$$\phi(\underline{x}) = \sum_{j=1}^M \phi'(\underline{y}_j). \quad (3.2)$$

Thus the class of multistate coherent structures specified by Barlow and Wu [2] (hereafter referred to as the BW class) is very closely related to the class of binary coherent structures. Exploiting this relationship makes it easy to extend results from the binary class to the BW class.

The axiomatic approach has proven to be more fertile in the multistate case. Axioms (i), (ii), and (iii) for the binary coherent structures can be generalized in a number of different ways each leading to a distinct class of multistate structures. We first survey the treatments in which the state space S is taken to be the set $\{0, 1, \dots, M\}$, representing $M+1$ levels of performance ranging from complete failure (0) to perfect functioning (M). The first class of multistate systems of this type is presented in EPS [6], where the structure function $\phi: S^n \longrightarrow S$ is assumed to satisfy three conditions.

3.1 Definition. A system of n components is said to be a multistate coherent system (MCS) if its structure function ϕ satisfies:

- (i)' ϕ is increasing.
- (ii)' For level j and component i , there exists a vector (\cdot, \underline{x}) such that $\phi(j, \underline{x}) = j$ while $\phi(l, \underline{x}) \neq j$ for $l \neq j$, $i = 1, \dots, n$ and $j = 0, \dots, M$.
- (iii)' $\phi(j) = j$ for $j = 0, 1, \dots, M$.

Note that conditions (i)' and (ii)' generalize conditions (i) and (ii) in

the binary case. Condition (iii)' is automatically satisfied in the binary case, but is not implied by (i)' and (ii)' in the present multistate case. The class of MCS's is referred to hereafter as the EPS class. It can be easily shown that the EPS class contains the BW class. In fact the EPS class is a much larger class. For instance, for a two component system, the BW class consists of two distinct systems only, namely the parallel system and the series system, regardless of the cardinality of S . However, for $M=2$ there are 12 structures in the EPS class.

In Definition 3.1, condition (ii) is referred to as the relevance condition for the components of the system. This leads to a type of coherence which is called by Griffith [9] strong coherence. The following two successively weaker types of relevancy are introduced by Griffith [9]:

(ii)'' For any component i and state $j \geq 1$, there exists a vector (\cdot_i, \underline{x}) such that $\phi((j-1)_i, \underline{x}) < \phi(j_i, \underline{x})$.

(ii)''' For any component i , there exists a vector (\cdot_i, \underline{x}) such that $\phi(o_i, \underline{x}) < \phi(M_i, \underline{x})$.

We now define the two new classes of multistate structures introduced by Griffith [9].

3.2 Definition. A structure function $\phi: S^n \longrightarrow S$ is said to be coherent (weakly coherent) if it satisfies conditions (i)', (ii)'' ((ii)'''), and (iii)'.

We denote by G1 class (G2 class) the class of coherent (weakly coherent) structures. It is easy to see that $BW \text{ class} \subseteq EPS \text{ class} \subseteq G1 \text{ class} \subseteq G2 \text{ class}$. Later in this section we demonstrate by examples that these classes are successively larger.

3.3 Remark. Conditions $(ii)'$ and $(ii)''$ describe successively weaker forms of relevancy of each component of the system to each level of performance. Condition $(ii)'''$ however, indicates only that the structure function is not constant in any of its arguments. In the binary case, the three conditions are equivalent.

In a recent paper by Natvig [13], the author introduces two classes of multistate systems called type 1 class and type 2 class (hereafter referred to as N1 class and N2 class respectively). The author introduces the following relevancy axiom which is weaker than $(ii)'$ but stronger than $(ii)''$:

$(ii)''''$ For any component i and state $j \geq 1$, there exists a vector (\cdot, \underline{x}) such that $\phi(j, \underline{x}) \geq j$ and $\phi((j-1), \underline{x}) \leq j-1$.

We now define the N1 class.

3.4 Definition. The N1 class is the class of all structure functions ϕ which satisfy conditions $(i)'$, $(ii)''''$, and $(iii)'$.

Note that condition $(ii)''''$ expresses the relevancy of each component i of the system to each state $j \geq 1$. Obviously $\text{EPS class} \subseteq \text{N1 class} \subseteq \text{G1 class}$

The binary coherent structure is then used by Natvig [13] to introduce yet another class of multistate systems.

3.5 Definition. The N2 class is the class of all structure functions

$\phi: S^n \longrightarrow S$ which has the representation $\phi(\underline{x}) = \sum_{j=1}^M \phi_j(I_j(\underline{x}))$, where $\phi_1 \geq \phi_2 \geq \dots \geq \phi_M$ are M binary coherent functions and $I_j(\underline{x})$ is the binary vector whose i th component is 1 iff $x_i \geq j$, $j=1, \dots, M$; $i=1, \dots, n$.

Note that when $\phi_1 = \phi_M$ we get precisely the BW class. In general the N2 class is larger than the BW class. For instance, when $n=2$, $M=2$ the BW class has 2 systems and the N2 class has 3 systems. In fact Natvig [13] shows that for $n=2$ ($n=3$) there are $M+1$

$$\left[9 + (M-1)30 + \binom{M-1}{2}46 + \binom{M-1}{3}33 + \binom{M-1}{4}9 \right] \text{ systems in the N2 class.}$$

3.6 Remark. It can be easily shown that the N2 class \subseteq N1 class. However, no containment holds in general between the N2 class and the EPS class. We demonstrate this fact by examples which are given later in this section.

Another class of multistate structures is introduced by Borges and Rodrigues [5] which they call the A*-type class (hereafter referred to as BR class).

3.7 Definition. A structure function $\phi: S^n \longrightarrow S$ belongs to the BR class iff it satisfies the following conditions:

- (1) ϕ is increasing
- (2) $(\{0, M\}^n) = \{0, M\}$.
- (3) For every i , there exists a vector (\cdot_i, \underline{x}) such that

$$\phi(0_i, \underline{x}) < \phi(M_i, \underline{x})$$

Note that condition (3) is the weak relevancy axiom due to Griffith [9]. Obviously BW class \subseteq BR class. The examples given later in this section illustrate that some other containments are not possible in general. Later in this section we present the main result in Borges and Rodrigues [5], which is a characterizing property for the BW class within the BR class.

Finally, Block and Savits [4] introduce a large class of multistate structures generalizing the class of binary structures called monotone

structures. The authors call their class the class of monotone multistate systems (denoted by MMS). We refer to this class as the BS class.

3.8 Definition. A structure function $\phi: S^n \longrightarrow S$ belongs to the BS class iff it satisfies the following conditions:

- (a) ϕ is increasing
- (b) $\phi(\underline{0})=0$, $\phi(\underline{M})=M$.

Obviously $G2 \text{ class} \subseteq BS \text{ class}$ and $BR \text{ class} \subseteq BS \text{ class}$.

3.9 Remark. It should be noted that condition (b) of Definition 3.8 implies that the set of weakly relevant components is not empty. Thus if in addition to (a) and (b) we require $\phi(\underline{k})=k$, $k=1, \dots, M-1$, we get a structure function ϕ of order $0 < j < n$ which belongs to the $G2$ class.

We summarize general containments that exist among the various classes introduced:

$$BW \text{ class} \subseteq (EPS \text{ class}) \cap (N2 \text{ class}) \cap (BR \text{ class}),$$

$$(EPS \text{ class}) \cup (N2 \text{ class}) \subseteq N1 \text{ class} \subseteq G1 \text{ class} \subseteq G2 \text{ class} \subseteq BS \text{ class}.$$

We now illustrate by examples that some other containments among these classes are not possible in general.

3.10 Example. Let $n=2$, $M=2$. Let ϕ_1 be the binary parallel system and ϕ_2 the binary series system. Let $\phi(\underline{x}) = \phi_1(I_1(\underline{x})) + \phi_2(I_2(\underline{x}))$. Then $\phi \in N2 \text{ class}$, but $\phi \notin EPS \text{ class}$. Note that in this case

$$(N2 \text{ class}) \cap (EPS \text{ class}) = (BW \text{ class})$$

3.11 Example. Let $n=2$, $M=2$. Define ϕ by $\phi(0,0)=\phi(1,0)=\phi(0,1)=0$, $\phi(1,1)=1$, $\phi(2,2)=\phi(1,2)=\phi(2,1)=\phi(2,0)=\phi(0,2)=2$. Then $\phi \in EPS \text{ class}$ but $\phi \notin N2 \text{ class}$.

3.12 Example. Let $n=2$, $M=2$. Define ϕ by $\phi(0,0)=\phi(1,0)=\phi(0,1)=0$,
 $\phi(1,1)=\phi(2,1)=\phi(2,0)=\phi(0,2)=1$, $\phi(1,2)=\phi(2,2)$. Then $\phi \in G1$ class but
 $\phi \notin N1$ class.

3.13 Example. Let $n=2$, $M=2$. Define ϕ by $\phi(0,0)=\phi(1,0)=0$,
 $\phi(0,1)=\phi(1,1)=\phi(1,2)=\phi(0,2)=1$, $\phi(2,0)=\phi(2,1)=\phi(2,2)=2$. Then $\phi \notin G2$ class
but $\phi \notin G1$ class.

3.14 Example. Let ϕ be defined as in Example 3.10. Then $\phi \in N2$ class
but $\phi \notin BR$ class.

3.15 Example. Let $n=2$, $M=2$. Define ϕ by $\phi(0,0)=\phi(1,0)=\phi(0,1)=0$,
 $\phi(1,1)=\phi(2,0)=\phi(0,2)=1$, $\phi(2,2)=\phi(1,2)=\phi(2,1)=2$. Then $\phi \in EPS$ class but
 $\phi \notin BR$ class.

3.16 Example. Let $n=2$, $M=2$. Define ϕ by $\phi(0,0)=\phi(1,0)=\phi(0,1)=\phi(1,1)=0$,
 $\phi(1,2)=\phi(2,1)=\phi(2,0)=\phi(2,2)=2$. Then $\phi \in BR$ class, but $\phi \notin (EPS \text{ class}) \cup$
 $(N2 \text{ class})$.

Some examples above are taken from references cited in this paper while
some are new.

The definition given by Ross [15] for a multistate system is less
structured than any of the definitions presented above. The state space
is taken to be $[0, \infty)$ and the structure function ϕ is any increasing
function from $[0, \infty)^n$ into $[0, \infty)$. Ross [15] does not attempt to
investigate structural properties of his model; rather, he concentrates on
the stochastic properties of his model when observed either at a fixed point
in time or when observed at different points in time (dynamic models).
Results of this type will be surveyed in the next two sections.

In the remainder of this section we present various structural properties of the multistate classes discussed above. These properties extend well known results in the binary case [1, Ch. 1] to the more general multistate classes.

The following theorem in EPS [6] gives simple bounds on the performance of any structure function in their class.

3.17 Theorem. Let ϕ be a structure function in the EPS class of order n . Then

$$\min_{1 \leq i \leq n} x_i \leq \phi(\underline{x}) \leq \max_{1 \leq i \leq n} x_i \quad (3.3)$$

Theorem 3.17 states that a parallel system yields the best performance in the EPS class and a series system yields the worst. Using this theorem, EPS [6] establish probabilistic bounds on system reliability. To establish Theorem 3.17, we need only conditions (i)' and (iii)' and therefore the result is true for the G2 class also.

The following lemma in EPS [6] gives a decomposition identity useful in carrying out inductive proofs. It holds for any multistate structure.

3.18 Lemma. The following identity holds for any n -component structure function ϕ :

$$\phi(\underline{x}) = \sum_{j=0}^M \phi(j_i, \underline{x}) I_{[x_i=j]}, \quad \text{for } i = 1, \dots, n, \quad (3.4)$$

where

$$I_{[x_i=j]} = \begin{cases} 1 & \text{if } x_i = j \\ 0 & \text{o.w.} \end{cases}$$

As in the binary case, EPS [6], define a dual structure for each multistate structure.

3.19 Definition. Let ϕ be the structure function of a multistate system. The dual structure function ϕ^D is given by:

$$\phi^D(\underline{x}) = M - \phi(M - x_1, \dots, M - x_n). \quad (3.5)$$

It is shown by EPS [6] and Griffith [9] that their classes are closed under the formation of dual structures. It can be shown that all the multistate classes that have been discussed in the present survey possess the same property.

Design engineers have used the well known principle that redundancy at the component level is preferable to redundancy at the system level. This principle is translated by EPS [6] into mathematical form in (i) of the following theorem; (ii) is a dual result. Extension of these results to the class of coherent structures is developed by Griffith [9].

3.20 Theorem. Let ϕ be a structure function in the EPS class.

Then

$$(i) \quad \phi(\underline{x} \vee \underline{y}) \geq \phi(\underline{x}) \vee \phi(\underline{y}).$$

$$(ii) \quad \phi(\underline{x} \wedge \underline{y}) \leq \phi(\underline{x}) \wedge \phi(\underline{y}).$$

Equality holds in (i) ((ii)) for all \underline{x} and \underline{y} iff the system is parallel (series).

Parts (i) and (ii) of Theorem 3.20 are also proved by Barlow and Wu [2].

3.21 Remark. It should be noted that parts (i) and (ii) of Theorem 3.20 are true for any increasing structure function. The only non-trivial fact of the theorem is that equality in (i) ((ii)) for all \underline{x} and \underline{y} implies the system is parallel (series). To establish this implication, EPS [6]

use the relevancy axiom (ii)', but then Griffith [9] shows that the weaker axiom (ii)' is sufficient. The structure function in Example 3.13 is used by Griffith [9] to show that (ii)'' is not sufficient to establish such an implication.

In binary coherent structures the concepts of minimal path vectors and minimal cut vectors play a crucial role. In the theory of multistate structures, generalizations of these concepts have been sought by the various authors. The first analogue of such concepts is defined by EPS [6] in the following:

3.22 Definition. A vector \underline{x} is said to be a connection vector to level j if $\phi(\underline{x}) = j$, $j = 0, 1, \dots, M$.

3.23 Definition. A vector \underline{x} is said to be an upper critical connection vector to level j if $\phi(\underline{x}) = j$ and $\underline{y} < \underline{x}$ imply $\phi(\underline{y}) < j$, $j = 1, \dots, M$.

A lower critical connection vector to level j can be defined in a dual manner, $j = 0, \dots, M-1$.

The existence of such critical connection vectors is guaranteed by the conditions of Definition 3.1. For $j=1, \dots, M$, let $\underline{y}_1^j, \dots, \underline{y}_{n_j}^j$ be the upper critical connection vectors to level j , where $\underline{y}_r^j = (y_{1r}^j, \dots, y_{nr}^j)$, $1 \leq r \leq n_j$. The following theorem by EPS [6] expresses the state of any structure function in the EPS class using its upper critical connection vectors.

3.24 Theorem. Let ϕ be a structure function in the EPS class. Let $\underline{y}_1^j, \dots, \underline{y}_{n_j}^j$ be its upper critical connection vectors to level j , $j=1, \dots, M$. Then $\phi(\underline{x}) \geq j$ iff $\underline{x} \geq \underline{y}_\ell^t$ for some $j \leq t \leq M$ and some $1 \leq \ell \leq n_t$.

The above theorem is utilized by EPS [6] to establish bounds on the system performance distribution, as will be shown in the next section.

3.25 Remark. In the paper by Griffith [9] it is asserted that $\phi(\underline{x}) \geq j$ iff $\underline{x} \geq \underline{y}_\ell^j$ for some $1 \leq \ell \leq n_j$. This assertion is unfortunately incorrect.

A second analogue to the concepts of min path (cut) vectors closely related to those of EPS [6] is introduced by Block and Savits [4], Borges and Rodrigues [5], and Natvig [13]. This concept is defined in the following:

3.26 Definition. A vector \underline{x} is called an upper vector for level j of a structure function ϕ if $\phi(\underline{x}) \geq j$. It is called critical upper vector for level j if in addition $\underline{y} < \underline{x}$ implies $\phi(\underline{y}) < j$, $j=1, \dots, M$.

A lower (critical lower) vector for level j can be similarly defined.

The existence of such vectors for the classes of multistate structures introduced by these authors is guaranteed by the axioms defining their classes.

Let ϕ be a structure function of a multistate system. Let C_j, U_j be the sets of upper critical connection vectors to level j and critical upper vectors for level j , respectively, $j=1, \dots, M$. Then obviously $C_j \subseteq U_j$,

$U_j \subseteq \bigcup_{t=j}^M UC_t$, $j=1, \dots, M$. Also, the following theorem is immediate.

3.27 Theorem. Let ϕ be a structure function that belongs to the BS class. Then $\phi(\underline{x}) \geq j$ iff $\underline{x} \geq \underline{y}$ for some $\underline{y} \in U_j$, $j=1, \dots, M$.

This is utilized by Block and Savits [4] and Natvig [13] to establish bounds on the system performance distribution (clearly Natvig [13] states this theorem for the N1 class). This theorem is utilized by Block and Savits [4] to give a max min representation for the structure functions in the BS class. A characterization for the BW class within the BS class in terms of U_j , $j=1, \dots, M$, is obtained by Block and Savits [4]. Another characterization for the BW class within the BR class is obtained by Borges and Rodrigues [5]

using the concept of critical upper vectors. We first present the characterization due to Borges and Rodrigues [5].

3.28 Theorem. Let ϕ be a structure function that belongs to the BR class. Then $\phi \in$ BW class iff for each \underline{x} such that $\phi(\underline{x}) \geq j$ there exists a vector $\underline{z} \in \{0, j\}^n$ satisfying $\underline{z} \leq \underline{x}$ and $\phi(\underline{z}) = j, j=1, \dots, M$.

The property stated in Theorem 3.28 is called by Borges and Rodrigues [5] property P. For $n=2, M=2$ the authors indicate that property P characterizes the BW class within the EPS class. However an example is presented in Borges and Rodrigues [5] to show that for other values of n, M , property P does not characterize the BW class within the EPS class.

The following representation of a structure function $\phi \in$ BS class is immediate from Theorem 3.27:

$$\phi(\underline{x}) = \sum_{j=1}^M \left\{ \max_{\underline{y} \in U_j} \min_{1 \leq i \leq n} I_{y_i}(\underline{x}) \right\} \quad (3.6)$$

where $I_{y_i}(\underline{x}) = 1$ if $x_i \geq y_i$ and $I_{y_i}(\underline{x}) = 0$ otherwise.

To formulate the expression in (3.6) in a more set-theoretic form, Block and Savits [4] introduce the following:

For every $\underline{x} \in U_j$, let $U_j(\underline{x}) = \{(i, x_i) : x_i > 0\}, j=1, \dots, M$. For every \underline{x} let $\alpha(\underline{x})$ be the $M \cdot n$ binary vector defined by $\alpha(\underline{x}) = (\alpha_{ij}(\underline{x}), i=1, \dots, n; j=1, \dots, M)$, where $\alpha_{ij}(\underline{x}) = 1$ iff $x_i \geq j$. The following theorem due to Block and Savits [4] converts the representation in (3.6) to a set-theoretic form.

3.29 Theorem. Let ϕ be a structure function in the BS class. Then

$$\phi(\underline{x}) = \sum_{j=1}^M \max_{\underline{y} \in U_j} \min_{(i,k) \in U_j(\underline{y})} \alpha_{ik}(\underline{x}).$$

A similar representation using critical lower vectors is also obtained by Block and Savits [4].

The following theorem due to Block and Savits [4] gives a characterization for the BW class within the BS class.

3.20 Theorem. Let ϕ be a structure function in the BS class. Then $\phi \in$ BW class iff $U_j = jU_1$, $j=1, \dots, M$.

3.21 Remark. A careful inspection of the concepts of critical upper vectors for level j and upper critical connection vectors to level j for a structure function ϕ in the BW class shows that the two concepts coincide. Also for such structure functions, the vector $\underline{y} = (y_1, \dots, y_n)$ is an upper critical connection vector to level j iff $y_i = j$ for $i \in P_\ell$, $y_i = 0$ for $i \in P_\ell^c$, where P_ℓ is one of the minimal path sets associated with ϕ .

In light of this remark Theorem 3.20 is immediate.

Decomposition of a binary coherent system into modules is useful in analyzing complex systems. Such a modular decomposition can be easily extended to a multistate model. A question raised and answered by Griffith [9] is whether a "relevant" component within a "relevant" module is "relevant" in the system. The answer is yes if relevancy is defined in terms of conditions (ii)', (ii)''. However an example is given by Griffith [9] to show that this is not necessarily the case for weak relevancy (condition (ii)'''). Simply define $\gamma(x_1, x_2, x_3) = \phi(\phi(x_1, x_2), x_3)$ where ϕ is the structure function in Example 3.13.

A deterministic measure of the importance of component i in a binary system ϕ is given by $\frac{1}{2^{n-1}} \sum_{(\cdot, \underline{x})} [\phi(1_i, \underline{x}) - \phi(0_i, \underline{x})]$. Note that

$(1_i, \underline{x}) - (0_i, \underline{x}) = 1$ iff component i is crucial to the functioning of the system when the states of the remaining components are described by (\cdot_i, \underline{x}) . Various generalizations of such a concept in the multistate setting are possible. In the next section some of these generalizations are obtained as special cases of probabilistic measures of importance introduced by the various authors.

4. Stochastic Properties of Multistate Systems.

Having discussed some structural aspects of the various multistate systems, we now turn to their probabilistic aspects. In this section we survey important relationships between the stochastic performance of a system in a given class and the stochastic performance of its components, which are often assumed to be independent, but in some cases assumed to be only associated (see [1, Ch. 2] for a definition and basic properties).

Let X_i denote the random state of component i , $i = 1, \dots, n$. Let $\underline{X} = (X_1, \dots, X_n)$ be the random vector representing the states of components $1, \dots, n$. Then $\phi(\underline{X})$ is the random variable representing the state of the system. In the models described by Barlow and Wu [2], Block and Savits [4], EPS [6], Griffith [9], and Natvig [13], the random variables X_1, \dots, X_n and $\phi(\underline{X})$ assume their values in the state space $S = \{0, 1, \dots, M\}$, with

$$\begin{aligned} P[X_i = j] &= p_{ij} & , & & P[\phi(\underline{X}) = j] &= p_j \\ P[X_i \leq j] &= P_i(j) & , & & P[\phi(\underline{X}) \leq j] &= P(j), \end{aligned} \quad (4.1)$$

for $j = 0, 1, \dots, M$ and $i = 1, \dots, n$. $P_i(\cdot)$ ($P(\cdot)$) represents the performance distribution of component i (system). Clearly,

$$P_i(j) = \sum_{k=0}^j p_{ik}, \quad P_i(M) = 1,$$

for $i = 1, \dots, n$. Similar relationships hold for P . Let $h = E(\phi(\underline{X}))$; when the components are assumed to be independent, we may express h as follows: $h \equiv h(P_1(\cdot), \dots, P_n(\cdot))$ since h is a function of $P_1(\cdot), \dots, P_n(\cdot)$. Alternatively, we may express h as follows: $h \equiv h(\underline{p}_1, \dots, \underline{p}_n)$, where $\underline{p}_i \equiv (p_{i0}, \dots, p_{iM})$ for $i = 1, \dots, n$. In either case, EPS [6] call h the performance function of the system.

The next lemma due to EPS [6] is obtained by a straightforward conditioning argument. The lemma expresses the performance function of a system of n components in terms of performance functions of systems of $n-1$ components. Such a decomposition is useful in deriving properties of h and in carrying out a proof by induction. It should be noted that the lemma is true for any structure function ϕ .

4.1 Lemma. The following identity holds for h :

$$h(\underline{p}_1, \dots, \underline{p}_n) = \sum_{j=0}^M p_{ij} h(j_i, \underline{p}_1, \dots, \underline{p}_n), \quad i = 1, \dots, n, \quad (4.2)$$

where $h(j_i, \underline{p}_1, \dots, \underline{p}_n) = E \phi(j_i, \underline{X})$.

The following theorem due to EPS [6] shows that h is strictly increasing in each p_{ij} , $j > 0$. This property generalizes the well known property of h in the binary case.

4.2 Theorem. Let $h(\underline{p}_1, \dots, \underline{p}_n)$ be the performance function of a system $\phi \in$ EPS class. Let $0 < p_{ij} < 1$ for $i = 1, \dots, n$; $j = 0, 1, \dots, M$. Then $h(\underline{p}_1, \dots, \underline{p}_n)$ is strictly increasing in p_{ij} , $i = 1, \dots, n$; $j = 1, \dots, M$.

It should be noted that this theorem holds for the performance function of a multistate structure ϕ in the G1 class.

Properties of h as a function of $P_1(\cdot), \dots, P_n(\cdot)$ are also investi-

gated by Barlow and Wu [2], EPS [6], Griffith [9], and Natvig [13]. The following theorem due to EPS [6] shows that in their class the function $h(P_1(\cdot), \dots, P_n(\cdot))$ is increasing with respect to stochastic ordering. A similar result is proved by Barlow and Wu [2] for their subclass using a different proof. The same property is proved by Griffith [9] and Ross [15] for their models. In fact the only property needed to establish the result is the monotonicity of the multistate structure ϕ .

4.3 Theorem. Let $P_i(\cdot)$ and $P'_i(\cdot)$ be two performance distributions for component i , $i=1, \dots, n$. Assume $p_i(j) \geq p'_i(j)$ for $j=0, \dots, M$; $i=1, \dots, n$. Let $P(\cdot)$ ($P'(\cdot)$) be the corresponding system performance distribution. Then

$$\begin{aligned} (i) \quad & P(j) \geq P'(j) \quad \text{for } j=0, 1, \dots, M, \\ (ii) \quad & h(P_1(\cdot), \dots, P_n(\cdot)) \leq h(P'_1(\cdot), \dots, P'_n(\cdot)). \end{aligned} \tag{4.3}$$

Using Theorem 3.17, EPS [6] obtain the following simple bounds on $P(\cdot)$ and h in terms of $P_1(\cdot), \dots, P_n(\cdot)$:

$$\begin{aligned} \prod_{i=1}^n P_i(j) &\leq P(j) \leq 1 - \prod_{i=1}^n \bar{P}_i(j), \\ \sum_{j=1}^M \prod_{i=1}^n \bar{P}_i(j-1) &\leq h \leq \sum_{j=1}^M [1 - \prod_{i=1}^n P_i(j-1)], \end{aligned} \tag{4.4}$$

where $\bar{P}_i(j) = 1 - P_i(j)$.

The inequalities in (4.4) are easily extendable to the case where the components are associated. In the papers by Block and Savits [4] and Natvig [13], such extensions are presented.

The concept of upper connection critical vectors introduced by EPS [6] is exploited to obtain further bounds on $P(\cdot)$ and h . Let $y_1^j, \dots, y_{n_j}^j$ be

the upper critical connection vectors to level j , $j = 1, \dots, M$. Let A_r^j denote the event $[\underline{X} \geq \underline{y}_r^j]$, $r = 1, \dots, n_j$. By Theorem 3.24,

$$P[\phi(\underline{X}) \geq j] = P\left(\bigcup_{t=j}^M \bigcup_{r=1}^{n_t} A_r^t\right).$$

Now using the well known inclusion-exclusion principle the authors establish upper and lower bounds on $P[\phi(\underline{X}) \geq j]$. Note that

$$P(A_r^j) = P[\underline{X} \geq \underline{y}_r^j] = \prod_{i=1}^n P[X_i \geq y_{ir}^j] \quad \text{for } 1 \leq r \leq n_j \text{ and } j = 1, \dots, M, \text{ where}$$

$$P[X_i \geq y_{ir}^j] = \sum_{\ell=y_{ir}^j}^M p_{i\ell}. \quad \text{Also since } h = \sum_{j=1}^M P[\phi(x) \geq j], \text{ bounds on } h \text{ can also be obtained.}$$

The concepts of critical upper (lower) vectors to level j , $j=1, \dots, M$, have been exploited by Block and Savits [4] and Natvig [13] to establish bounds on $P(\cdot)$ and h . For example Natvig [13] proves the following:

4.4 Theorem. Let ϕ be a multistate structure in the N1 class. Let

$\underline{y}_1^j, \dots, \underline{y}_{n_j}^j$ be critical upper vectors to level j , $j = 1, \dots, M$. Let

$C_1^j(\underline{y}_r^j) = \{i: y_{ir}^j > 0\}$, $r = 1, \dots, n_j$; $j = 1, \dots, M$. Assume the components are associated. Then

$$P[\phi(x) \geq j] \leq 1 - \prod_{r=1}^{n_j} [1 - P(\bigcup_{i \in C_1^j(\underline{y}_r^j)} \{X_i \geq y_{ir}^j\})].$$

Similar bounds are obtained by Block and Savits [4] using their representation in Theorem 3.29.

An interesting generalization of the Moore-Shannon Theorem [1, Chap. 2, Theorem 5.4] is obtained by Barlow and Wu [2]. In view of (3.2), it is easily verified that

$$P[\phi(\underline{X}) \geq j] = E \phi'(\underline{Y}_j) = h'(\underline{q}_j), \quad (4.5)$$

where $\underline{q}_j = (q_{1j}, \dots, q_{nj})$ and $q_{ij} = \sum_{k=j}^M p_{ik}$, $i = 1, \dots, n$.

Recall that Moore and Shannon show that if all components have the same reliability p , then either $h(p) \geq p$ or $h(p) \leq p$ for all $0 \leq p \leq 1$, or there exists $0 < p_0 < 1$ such that $h(p) \leq p$ for $0 \leq p \leq p_0$, while $h(p) \geq p$ for $1 \geq p \geq p_0$. Barlow and Wu [2] give a natural generalization of this result to the multistate case with respect to stochastic ordering.

4.5 Theorem. Let $\underline{p}_i = \underline{\alpha} = (\alpha_0, \dots, \alpha_M)$ for $i = 1, \dots, n$. Assume $h'(p_0) = p_0$ ($0 < p_0 < 1$). Let $\alpha^* = (1-p_0, 0, \dots, 0, p_0)$. Then

$$\begin{array}{ll} \underline{\alpha} \leq \underline{\alpha}^* \text{ st} & \text{implies that } \underline{p} \leq \underline{\alpha}, \\ \underline{\alpha} \leq \underline{\alpha}^* \text{ st} & \text{implies that } \underline{p} \geq \underline{\alpha}, \end{array}$$

where $\underline{p} = (p_0, \dots, p_M)$, $p_i = P[\phi(\underline{X}) = i]$, $i = 0, \dots, M$.

Note that (4.5) is central to the proof of this theorem.

In the model proposed by Ross [15], X_i , $i = 1, \dots, n$, and $\phi(\underline{X})$ are nonnegative random variables with distribution functions F_i , $i = 1, \dots, n$, and F respectively. The function $r(\bar{F}_1, \dots, \bar{F}_n)$ is defined by $r(\bar{F}_1, \dots, \bar{F}_n) = E \phi(\underline{X})$.

Using an extension of Lemma 2.3, Chap. 4, of Barlow and Proschan [1], Ross [15] proves the following:

4.6 Theorem. If ϕ is a binary increasing function then

$$r(\bar{F}_1^\alpha, \dots, \bar{F}_n^\alpha) \geq [r(\bar{F}_1, \dots, \bar{F}_n)]^\alpha \quad (4.6)$$

for all $0 \leq \alpha \leq 1$.

As a consequence of this theorem, Ross [15] proves:

4.7 Corollary. Let X_1, \dots, X_n be independent IFRA random variables. Then

$$(a) \quad \sum_{i=1}^n X_i \text{ is IFRA,}$$

$$(b) \quad P\left[\prod_{i=1}^n X_i > a \alpha^n\right] \geq (P\left[\prod_{i=1}^n X_i > a\right])^\alpha, \quad 0 \leq \alpha \leq 1.$$

Observe that part (a) of Corollary 4.7 represents the well known property of the closure of the IFRA distribution under the convolution operation.

Finally, several authors suggest generalizations of the concept of the reliability importance of component i which is defined in the binary case by $I(i) = P[\phi(0_i, \underline{X}) < \phi(1_i, \underline{X})]$. The following importance measures are due to Barlow and Wu [2], Block and Savits [4], and Natvig [13]:

$$I_j^1(i) = P[\phi(j_i, \underline{X}) = j, \phi(l_i, \underline{X}) \neq j, l \neq j],$$

$$I_j^2(i) = P[\phi((j-1)_i, \underline{X}) < \phi(j_i, \underline{X})],$$

$$I_j^3(i) = P[\phi(0_i, \underline{X}) < \phi(M_i, \underline{X})].$$

Note that for various classes one must choose the appropriate measure. Taking $P[X_i=j] = \frac{1}{M+1}$, $j=0, \dots, M$, the above measures are converted to structural importance measures of the various components. In Griffith [9], the importance of component i is measured by a vector. Let $a_M \geq a_{M-1} \geq \dots \geq a_0 = 0$ be utilities associated with the various levels of performance of the system.

The expected utility of the system is $\sum_{i=1}^M a_i P[\phi(\underline{X}) = i] = \sum_{i=1}^M b_i P[\phi(\underline{X}) \geq i]$

where $b_1 = a$, $b_k = a_k - a_{k-1}$, $k = 2, \dots, M$. Define $I_{lj}(i) =$

$P[\phi(l_i, X) \geq j] - P[\phi((l-1)_i, X) \geq j]$, $i, j, l \geq 1$. Let $I_l(i) = \sum_{j=1}^M b_j I_{lj}(i)$, $l = 1, \dots, M$. Let $I(i) = (I_1(i), \dots, I_M(i))$, then the vector $I(i)$ is called by Griffith [9] the importance vector of component i . Relationships between the expected utility of the system and the importance vectors of the components are presented by Griffith [9]. For example he shows that if the expected utility is viewed as a function of the i^{th} component distribution, keeping all the other marginals fixed for $j \neq i$, then

$$I(i) = \text{grad}_i u(\underline{p}_1, \dots, \underline{p}_n), \quad i = 1, \dots, n,$$

where $\underline{p}_i = (p_{i1}, \dots, p_{iM})$ and $p_{il} = P[X_i \geq l]$, $i = 1, \dots, n$; $l = 1, \dots, M$.

5. Dynamic Models for Multistate Coherent Systems.

In the binary reliability models, the length of time during which a component or system functions is called the lifelength of the component or system; these lifelengths are nonnegative random variables. Classes of lifelength distributions based on various notions of aging have been introduced and studied. See, e.g., [1]. Two of the important classes of life distributions are the increasing failure rate average (IFRA) class and the new better than used (NBU) class. Closure of these classes under basic reliability operations, such as convolution of distributions and formation of coherent systems, have been established. The counterparts of these concepts in the multistate case have been first investigated by Barlow and Wu [2], EPS [6], and Ross [15]. More recently, Block and Savits [3] and El-Newehi [8] introduced general multivariate versions of these concepts.

Let $\{X_i(t), t \geq 0\}$ denote the decreasing and right continuous stochastic process representing the state of component i at time t , where t ranges over the nonnegative real numbers for $i = 1, \dots, n$. The processes $\{X_i(t), t \geq 0\}$, $i = 1, \dots, n$, are assumed to be mutually independent. The stochastic process $\{\phi(\underline{X}(t)), t \geq 0\}$ is also decreasing and right continuous and represents the corresponding system state as time varies, where $\underline{X}(t) = (X_1(t), \dots, X_n(t))$, $t \geq 0$.

In the model of Barlow and Wu [2] the state space is $\{0, 1, \dots, M\}$. Let us call $\{j, j+1, \dots, M\}$ the "good" states. Assume that $[P(X_i(t) \geq j)]^{1/t}$ is decreasing in $t \geq 0$ for fixed j , $i = 1, \dots, n$. It is easily verified that $[P(\phi(\underline{X}(t)) \geq j)]^{1/t}$ is decreasing in $t \geq 0$ for fixed j . Thus the above result states that if the length of time spent by each component in the "good" states is an IFRA random variable, then the corresponding length of time spent by the multistate system in the "good" states is also an IFRA random variable. In the binary case this represents the so-called IFRA closure (under formation of binary coherent systems) theorem.

The following definition is due to Ross [15].

5.1 Definition. The stochastic process $\{X(t), t \geq 0\}$ is said to be an IFRA process if $T_a = \inf\{t: X(t) \leq a\}$ is an IFRA random variable for every $a \geq 0$.

Having introduced this definition, Ross [15] then proves the following generalized IFRA closure theorem.

5.2 Theorem. Let $\{X_i(t), t \geq 0\}$, $i = 1, \dots, n$, be independent IFRA processes and ϕ a multistate structure function. Then $\{\phi(\underline{X}(t)), t \geq 0\}$ is an IFRA process.

The crucial tool in proving this theorem is Theorem 4.6.

Ross [15] also defines an NBU process and proves a generalized NBU closure theorem (under formation of multistate structures).

A different definition of an NBU process is given by EPS [6], and then a simple characterization for this NBU process is derived. Using their characterization, they give a simple proof of a generalized NBU closure theorem. The EPS definition of an NBU process is as follows:

5.3 Definition. The stochastic process $\{X_i(t), t \geq 0\}$ is an NBU process if $T_{i,j} = \inf\{t: X_i(t) \leq j\}$ is an NBU random variable for $j = 0, \dots, M$ and $i = 1, \dots, n$.

Recall that the state space for the EPS [6] model is the set $\{0, \dots, M\}$.

The following lemma gives a simple characterization of an NBU process.

5.4 Lemma. The stochastic process $\{X_i(t), t \geq 0\}$ is NBU if and only if for all $s \geq 0, t \geq 0$,

$$X_i(s+t) \stackrel{st}{\leq} \min(X_i'(s), X_i'(t)),$$

where $X_i'(s)$ and $X_i'(t)$ are two independent random variables having the same distributions as $X_i(s), X_i(t)$ respectively.

Using their Lemma 5.1, EPS [6], prove the following generalized NBU closure theorem.

5.5 Theorem. Let ϕ be a structure function in the EPS class having n components and $\{X_i(t), t \geq 0\}$ be the i^{th} component performance process, $i = 1, \dots, n$. Let $\{X_i(t), t \geq 0\}, i = 1, \dots, n$, be independent NBU processes. Then $\{\phi(X(t)), t \geq 0\}$ is an NBU stochastic process.

The various generalizations that have been presented so far in this section have been obtained under the assumption that the components of the system are independent. However in many real life situations the components are subjected to common stresses which make them stochastically dependent. In recent papers by Block and Savits [3] and El-Newehi [8], the authors introduce multivariate classes of stochastic processes that describe the joint performance of the n components of a system without requiring statistical independence of components.

Now let $\{X(t) \equiv (X_1(t), \dots, X_n(t), t \geq 0)\}$ be a vector-valued stochastic process. Assume that $X(t)$ is nonnegative, decreasing and right-continuous.

The following definition is due to Block and Savits [3].

5.6 Definition. $\{X(t), t \geq 0\}$ is said to be a vector-valued IFRA process if and only if for every upper open set $U \subset R^n$, the random variable

$$T_U = \inf \{t: X(t) \notin U\}$$

is IFRA.

Block and Savits prove the following closure theorem:

5.7 Theorem. If ϕ is a multistate monotone structure function and $\{X(t), t \geq 0\}$ is an IFRA process, then $\{\phi(X(t)), t \geq 0\}$ is an IFRA process.

The following definition is due to El-Newehi [8].

5.8 Definition. The vector-valued stochastic process $\{X(t), t \geq 0\}$ is said to be an MNBU process if and only if the random variable

$$T_C = \inf \{t: X(t) \in C\}$$

is NBU for every lower closed set $C \subseteq R^n$.

The following generalized NBU closure theorem is then proved by El-Newehi [8].

5.9 Theorem. Let $\{X(t), t \geq 0\}$ be an MNBU process. Let ϕ be a decreasing, left-continuous, nonnegative function. Then $\{\phi(X(t)), t \geq 0\}$ is an NBU process.

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